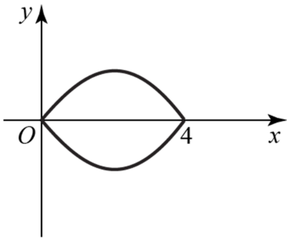
Exercise 5B

1



From symmetry the centre of mass lies on the x-axis.

As $y = x^2 - 4x$ meets the x-axis when

$$x^2 - 4x = 0$$

$$\text{i.e. } x(x-4) = 0$$

$$\therefore \quad x = 0 \text{ and } x = 4$$

Again from symmetry the centre of mass lies at $x = 2$.

The coordinates of the centre of mass are (2, 0).

2 The curve with equation

$$(x-1)^2 + y^2 = 1$$

is a circle, centre (1, 0) radius 1.

It is rotated about the x-axis to form a sphere – centre (1, 0).

The centre of mass is at (1, 0).

3 By symmetry, $\bar{y} = 0$. Using

$$\bar{x} = \frac{\int_{\pi/2}^{3\pi/2} xy^2 \, dx}{\int_{\pi/2}^{3\pi/2} y^2 \, dx} = \frac{\int_{\pi/2}^{3\pi/2} x\cos^2 x \, dx}{\int_{\pi/2}^{3\pi/2} \cos^2 x \, dx}$$

$$= \frac{\int_{\pi/2}^{3\pi/2} \frac{1}{2}x(1+\cos 2x) \, dx}{\int_{\pi/2}^{3\pi/2} \frac{1}{2}(1+\cos 2x) \, dx} = \frac{\int_{\pi/2}^{3\pi/2} x(1+\cos 2x) \, dx}{\left[x + \frac{1}{2}\sin 2x\right]_{\pi/2}^{3\pi/2}}$$

$$= \frac{\left[\frac{1}{2}x^2\right]_{\pi/2}^{3\pi/2} + \int_{\pi/2}^{3\pi/2} x\cos 2x \, dx}{\pi}$$

$$= \pi + \frac{1}{\pi}\int_{\pi/2}^{3\pi/2} x\cos 2x \, dx$$

Note that by symmetry, the remaining integral will evaluate to zero.

Hence $\bar{x} = \pi$.

Hence the centre of mass is at $(\pi, 0)$.

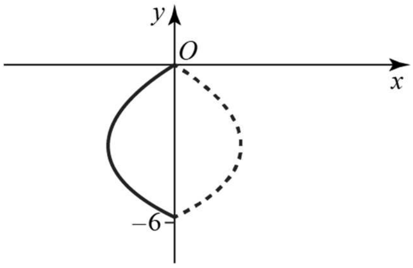
4 The curve with equation $y^2+6y=x$ meets the x-axis at $x = 0$, and meets the y-axis when

$y^2+6y=0$

i.e. $y(y+6)=0$

$\therefore\ y=0 \text{ or } -6$

The curve is shown in the diagram, and is rotated about the y-axis through 360°.



From symmetry the centre of mass lies at the point (0,–3).

5 The centre of mass lies on the x-axis, from symmetry.

Using the formula

$$\bar{x}=\frac{\int \pi y^2 x\,\mathrm{d}x}{\int \pi y^2\,\mathrm{d}x}, \text{as } y=3x^2$$

$$\bar{x}=\frac{\int_0^1 \pi\times 9x^5\,\mathrm{d}x}{\int_0^1 \pi\times 9x^4\,\mathrm{d}x}$$

$$=\frac{\pi\left[\frac{9}{6}x^6\right]_0^1}{\pi\left[\frac{9}{5}x^5\right]_0^1}$$

$$=\frac{9}{6}\div\frac{9}{5}$$

$$=\frac{5}{6}$$

$\therefore$ The centre of mass lies at the point $\left(\frac{5}{6},0\right)$

6 The centre of mass lies on the x-axis, from symmetry.

Using the formula $\bar{x} = \dfrac{\int \pi y^2 x \, dx}{\int \pi y^2 \, dx}$, with $y = \sqrt{x}$

$$\bar{x} = \frac{\int_0^4 \pi x \times x \, dx}{\int_0^4 \pi x \, dx} = \frac{\int_0^4 \pi x^2 \, dx}{\int_0^4 \pi x \, dx}$$

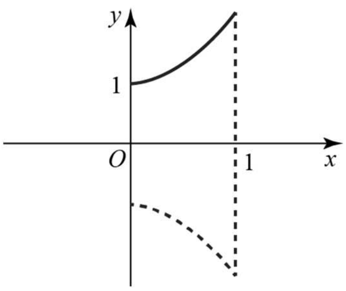
then $= \dfrac{\left[\frac{1}{3}x^3\right]_0^4}{\left[\frac{1}{2}x^2\right]_0^4}$

$$= \frac{64}{3} \div 8$$

$$= \frac{8}{3}$$

$\therefore$ The centre of mass lies at the point $\left(\frac{8}{3}, 0\right)$

7



The centre of mass lies on the x-axis from symmetry.

Using the formula $\bar{x} = \dfrac{\int \pi y^2 x \, dx}{\int \pi y^2 \, dx}$, as $y = 3x^2 + 1$,

$$\bar{x} = \frac{\int_0^1 \pi\left(3x^2+1\right)^2 x \, dx}{\int_0^1 \pi\left(3x^2+1\right)^2 \, dx}$$

$$= \frac{\pi\int_0^1 9x^5 + 6x^3 + x \, dx}{\pi\int_0^1 9x^4 + 6x^2 + 1 \, dx}$$

$$= \frac{\left[\frac{9}{6}x^6 + \frac{6}{4}x^4 + \frac{1}{2}x^2\right]_0^1}{\left[\frac{9}{5}x^5 + \frac{6}{3}x^3 + x\right]_0^1}$$

$$= \frac{\frac{9}{6} + \frac{6}{4} + \frac{1}{2}}{\frac{9}{5} + \frac{6}{3} + 1} = \frac{3\frac{1}{2}}{4\frac{4}{5}}$$

$$= \frac{35}{48}$$

$\therefore$ The centre of mass lies at the point $\left(\frac{35}{48}, 0\right)$

8 The centre of mass lies on the x-axis from symmetry.

Using the formula $\bar{x} = \dfrac{\int \pi y^2 x\,\mathrm{d}x}{\int \pi y^2\,\mathrm{d}x}$, with $y = \dfrac{3}{x}$, then

$$\bar{x} = \frac{\int_1^3 \pi\left(\frac{9}{x^2}\right)x\,\mathrm{d}x}{\int_1^3 \pi\left(\frac{9}{x^2}\right)\mathrm{d}x}$$

$$= \frac{\pi\int_1^3 \frac{9}{x}\,\mathrm{d}x}{\pi\int_1^3 9x^{-2}\,\mathrm{d}x}$$

$$= \frac{[9\ln x]_1^3}{\left[-9x^{-1}\right]_1^3}$$

$$= \frac{9\ln 3}{9-3}$$

$$= \frac{3}{2}\ln 3$$

$\therefore$ The centre of mass lies at the point $\left(\frac{3}{2}\ln 3, 0\right) = (1.65, 0)$ (3 s.f.)

9 The centre of mass lies in the x-axis from symmetry.

Using the formula $\bar{x} = \dfrac{\int \pi y^2 x\,\mathrm{d}x}{\int \pi y^2\,\mathrm{d}x}$, with $y = 2\mathrm{e}^x$, then

$$\bar{x} = \frac{\int_0^1 \pi \times 4\mathrm{e}^{2x} \times x\,\mathrm{d}x}{\int_0^1 \pi \times 4\mathrm{e}^{2x}}$$

$$= \frac{2\pi\left\{\left[x\mathrm{e}^{2x}\right]_0^1 - \int_0^1 \mathrm{e}^{2x}\,\mathrm{d}x\right\}}{2\pi\int_0^1 2\mathrm{e}^{2x}\,\mathrm{d}x}$$

$$= \frac{\left[x\mathrm{e}^{2x} - \frac{1}{2}\mathrm{e}^{2x}\right]_0^1}{\left[\mathrm{e}^{2x}\right]_0^1}$$

$$= \frac{\mathrm{e}^2 - \frac{1}{2}\mathrm{e}^2 + \frac{1}{2}}{\mathrm{e}^2 - 1}$$

$$= \frac{1}{2}\frac{(\mathrm{e}^2+1)}{(\mathrm{e}^2-1)}$$

$\therefore$ The centre of mass lies at the point $\left(\dfrac{(\mathrm{e}^2+1)}{2(\mathrm{e}^2-1)}, 0\right)$

10 By symmetry, $\bar{x} = 0$. Using $x = 3\,\mathrm{e}^{-y}$,

$$\bar{y} = \frac{\int_1^2 yx^2\,\mathrm{d}y}{\int_1^2 x^2\,\mathrm{d}y} = \frac{\int_1^2 9y\mathrm{e}^{-2y}\,\mathrm{d}y}{\int_1^2 9\mathrm{e}^{-2y}\,\mathrm{d}y}$$

$$= \frac{\left[-\tfrac{1}{2}y\mathrm{e}^{-2y}\right]_1^2 + \tfrac{1}{2}\int_1^2 \mathrm{e}^{-2y}\,\mathrm{d}y}{\left[-\tfrac{1}{2}\mathrm{e}^{-2y}\right]_1^2}$$

$$\approx \frac{0.04935 + 0.5 \times 0.05851}{0.05851} \approx 1.34 \text{ (3 s.f.)},$$

where we used integration by parts.

$\therefore$ The centre of mass lies at the point (0,1.34).

11 By symmetry, $\bar{y} = 0$. Using $y = \dfrac{2}{1+x}$

$$\bar{x} = \frac{\int_0^4 xy^2\,\mathrm{d}x}{\int_0^4 y^2\,\mathrm{d}x} = \frac{\int_0^4 4x(1+x)^{-2}\,\mathrm{d}x}{\int_0^4 4(1+x)^{-2}\,\mathrm{d}x}$$

$$= \frac{\int_0^4 x(1+x)^{-2}\,\mathrm{d}x}{\left[-(1+x)^{-1}\right]_0^4} = \frac{5}{4}\int_0^4 \frac{x}{(1+x)^2}\,\mathrm{d}x$$

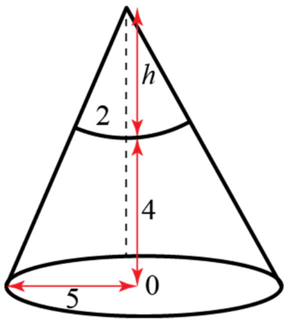
$$= \frac{5}{4}\int_0^4 \left(\frac{1}{x+1} - \frac{1}{(x+1)^2}\right)\mathrm{d}x$$

$$= \frac{5}{4}\left[\ln(x+1)\right]_0^4 - \frac{5}{4}\left[-(x+1)^{-1}\right]_0^4$$

$$= \frac{5}{4}\ln 5 - 1 \approx 1.01 \text{ (3 s.f.)}$$

Here for integration we used the method of partial fractions.

12 A frustum of a cone is obtained by removing a small cone from a large cone.
Draw a diagram, showing the cones and the frustum and let the height of the small cone be h cm.



Using similar triangles:

$$\frac{h}{h+4}=\frac{2}{5}$$

$$\therefore\ 5h=2(h+4)$$

$$\therefore\ 3h=8$$

$$\text{i.e. } h=\frac{8}{3}$$

The centre of mass lies on the axis of symmetry. Let the centre of the base of the large cone be O.

Shape	Mass	Units of mass	Distance of centre of mass from O
Large cone	$\rho\times\frac{1}{3}\pi\times5^2\times\frac{20}{3}$	500	$\frac{5}{3}$
Small cone	$\rho\times\frac{1}{3}\pi\times2^2\times\frac{8}{3}$	32	$4+\frac{2}{3}$
Frustum	$\frac{500\pi}{9}\rho-\frac{32\pi}{9}\rho$	468	$\bar{x}$

Take moment about O

$$500\times\frac{5}{3}-32\times\left(\frac{14}{3}\right)=468\bar{x}$$

$$\therefore\ \frac{2052}{3}=468\bar{x}$$

$$\therefore\ \bar{x}=\frac{19}{13}\text{ or } 1.46\ (3\text{ s.f.})$$

13 First let us find the centre of mass of the frustum. Considering the frustum as a part of a cone, the height of the larger cone (using similar triangles) is

$$\frac{H-8}{H}=\frac{2}{4}\Rightarrow H=16$$

Denoting $h=H-8$ and taking moments about the base of the frustum

$$\tfrac{1}{3}\rho\pi4^2H\times\tfrac{1}{4}H-\tfrac{1}{3}\rho\pi2^2h\times(\tfrac{1}{4}h+8)$$

$$=(\tfrac{1}{3}\rho\pi4^2H-\tfrac{1}{3}\rho\pi2^2h)\bar{x}$$

From this equation, we obtain $\bar{x}=\frac{22}{7}$. The mass of the frustum is $M=\frac{224}{3}\pi\rho$, and the mass of the cylinder is $m=8\pi\rho$

Taking the moments about the base of the hollow frustum $M\bar{x}=(M-m)\bar{X}+4m$, where $\bar{X}$ is the centre of mass of the resulting solid. Solving this equation gives $\bar{X}=3.04$ cm.

14 Let the density of the material be ρ and its thickness be t.
Then the mass will be proportional to the surface area.
The centre of mass will be on the axis of symmetry

Shape	Mass	Units of mass	Distance of centre of mass from *O*
Hemispherical shell	$\rho t \times 2\pi r^2$	2	$\frac{r}{2}$
Circular base	$\rho t \times \pi r^2$	1	0
Composite body	$\rho t \times 3\pi r^2$	3	$\bar{x}$

Take moment about centre of base:

$$2 \times \frac{r}{2} + 0 = 3\bar{x}$$

$$\therefore \quad \bar{x} = \frac{1}{3}r$$

So the centre of mass is at a distance $\frac{r}{3}$ above the base.

15 Let the density of the material be ρ and its thickness be t.
Then the mass will be proportional to the surface area.
The centre of mass will be on the axis of symmetry

Shape	Mass	Units of mass	Distance of centre of mass above base
Hollow cone	$\rho t \times \pi \times 3 \times 5$	15	$\frac{1}{3} \times 4$
Circular base	$\rho t \times \pi \times 3^2$	9	0
Composite body	$15\pi\rho t + 9\pi\rho t$	24	$\bar{x}$

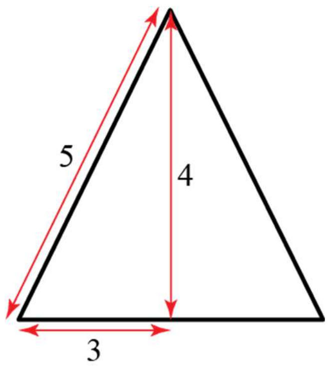
[The surface area of a cone is given by the formula $\pi r l$ where l is the length of the slant side.
As $r = 3$ and $h = 4$ then $l = 5$ from Pythagoras-theorem.]

Take moment about centre of base:

$$\therefore 15 \times \frac{4}{3} + 0 = 24\bar{x}$$

$$\therefore \bar{x} = \frac{20}{24} = \frac{5}{6}$$

So the centre of mass is at a distance $\frac{5}{6}$ cm above the base.



16 The centre of mass of the cap from the centre of the ball is

$$\bar{x} = \frac{\int_{a-h}^{a} xy^2\,dx}{\int_{a-h}^{a} y^2\,dx} = \frac{\int_{a-h}^{a} x(a^2-x^2)\,dx}{\int_{a-h}^{a} (a^2-x^2)\,dx}$$

$$= \frac{\left[\frac{1}{2}a^2x^2 - \frac{1}{4}x^4\right]_{a-h}^{a}}{\left[a^2x - \frac{1}{3}x^3\right]_{a-h}^{a}} = \frac{12a^2 - 12ah + 3h^2}{4(3a-h)}$$

The centre of mass from the base of the cap will be

$$\bar{x} - (a-h) = \frac{h(4a-h)}{4(3a-h)}$$

17 a The curve $y = 8 - x^3$ crosses the y-axis at $y = 8$. We also find that $x = (8-y)^{1/3}$.

Using $\bar{y} = \dfrac{\int_0^8 yx^2\,dy}{\int_0^8 x^2\,dy} = \dfrac{\int_0^8 y(8-y)^{2/3}\,dy}{\int_0^8 (8-y)^{2/3}\,dy}$

We will find these integrals separately. Firstly $\int_0^8 (8-y)^{2/3}\,dy = \left[-\frac{3}{5}(8-y)^{5/3}\right]_0^8 = \dfrac{96}{5}$

For the numerator, make the substitution $u = 8 - y$

$$\int y(8-y)^{2/3}\,dy = \int (u-8)u^{2/3}\,du$$

$$= \int u^{5/3}\,du - 8\int u^{2/3}\,du = \tfrac{3}{8}u^{8/3} - \tfrac{24}{5}u^{5/3}$$

$$= \tfrac{3}{8}(8-y)^{8/3} - \tfrac{24}{5}(8-y)^{5/3}$$

$= -\frac{3}{40}(8-y)^{5/3}(5y+24)$ and now we can find $\int_0^8 y(8-y)^{2/3}\,dy \approx \dfrac{288}{5}$ and $\bar{y} = 3$.

b The radius of the hemisphere is where the curve intersects the x-axis, at $x = 2$.
We will take moments about the base of the egg, giving $M\bar{Y} = m_1\left(\frac{5}{8}\times 2\right) + m_2(\bar{y}+2)$, where $\bar{Y}$ is the centre of mass of the egg from the base, M is the mass of the egg, m_1 is the mass of the solid hemisphere, and m_2 is the mass of the solid S. The mass of the solid is

$m_2 = \int_0^8 \pi\rho(8-y)^{2/3}\,dy \approx \dfrac{96\pi\rho}{5}$, and so

$$\bar{Y} = \frac{\frac{96}{5}\pi(\bar{y}+2) + \frac{2}{3}\pi 8\left(\frac{5}{8}\times 2\right)}{\frac{96}{5}\pi + \frac{2}{3}\pi 8} = \frac{385}{92}$$

≈ 4.18 cm (3 s.f.).

Here we also used the fact that the centre of mass of the solid hemisphere from its flat surface is $\frac{3}{8}$ times its radius.

18 Taking moments about the point O gives

$$\tfrac{1}{3}\pi r^2 kx \times \tfrac{1}{4}kx - \tfrac{1}{3}\pi r^2 x \times \tfrac{1}{4}x = \left(\tfrac{1}{3}\pi r^2 kx + \tfrac{1}{3}\pi r^2 x\right)\tfrac{1}{10}x$$

where we used the fact that the centre of mass of a solid cone lies $\frac{1}{4}$ times its height above its flat base. Solving the equation gives $k = 1.4$.

19 a The mass of the cylinder is $m_1 = \pi(2r)^2 h\rho = 4\pi r^2 h\rho$, and the mass of the cone is $m_2 = \frac{1}{3}\pi r^2 h\rho$.

The resulting solid is symmetric about the vertical plane going through the axes of symmetry of the cone and the cylinder, thus the centre of mass will lie in that plane. Taking moments in the vertical direction about the top face,

$$m_1 \tfrac{1}{2}h - m_2 \tfrac{1}{4}h = (m_1 - m_2)kh$$

$$4\pi r^2 h \tfrac{1}{2}h - \tfrac{1}{3}\pi r^2 h \tfrac{1}{4}h = (4\pi r^2 h - \tfrac{1}{3}\pi r^2 h)kh$$

This gives $k = \frac{23}{44}$

b Taking moments about point O in the horizontal direction,

$$m_1 \times 0 - m_2 r = -(m_1 - m_2)\tilde{k}r \Rightarrow$$

$$\tilde{k} = \frac{m_2}{m_1 - m_2} = \frac{\frac{1}{3}\pi r^2 h}{4\pi r^2 h - \frac{1}{3}\pi r^2 h} = \frac{1}{11}$$

Hence the centre of mass of the solid is $\frac{1}{11}r$ away from the axis of the cylinder, away from the cavity, on the axis of symmetry.

20 a Taking moments about the base of the cylinder

$$\tfrac{1}{3}\pi \times 5^2 \times (\tfrac{1}{4} \times 12 + 3) + \pi \times 5^2 \times 3 \times \tfrac{1}{2} \times 3$$

$= (\frac{1}{3}\pi \times 5^2 + \pi \times 5^2 \times 3)\bar{y}$. This gives $\bar{y} = \frac{57}{14} \approx 4.07$ cm (3 s.f.).

b If the game piece is hollow, the mass will be proportional to the area. The area of the sides of the cylinder is $2\pi 5 \times 3 = 30\pi$, and the area of the cone shell is $\pi 5(5 + \sqrt{5^2 + 12^2}) = 90\pi$.

Taking moments about the base of the cylinder

$$90\pi\left(\frac{1}{3} \times 12 + 3\right) + 30\pi\frac{1}{2} \times 3$$

$= \left(90\pi + 30\pi + \pi 5^2\right)\bar{y}$, which gives

$\bar{y} = \frac{135}{29} \approx 4.66$ cm (3 s.f.)

Challenge

The curved surface area of the elemental disc is $2\pi NPR\,\delta\theta = 2\pi R\sin\theta \times \delta\theta$

$\therefore$ its mass is $2\pi\rho R^2 \sin\theta\delta\theta$, where ρ is the mass per unit area.

$\therefore$ Since $\sum m_i \times \bar{x} = \sum m_i x_i$, where $x_i = R\cos\theta$

and $m_i = 2\pi\rho R^2 \sin\theta\,\delta\theta$ then

$$\bar{x} = \frac{\int_0^{\frac{\pi}{2}} 2\pi\rho R^2 \sin\theta \times R\cos\theta\,\mathrm{d}\theta}{\int_0^{\frac{\pi}{2}} 2\pi\rho R^2 \sin\theta\,\mathrm{d}\theta}$$

$$= \frac{R\int_0^{\frac{\pi}{2}} \sin\theta\cos\theta\,\mathrm{d}\theta}{\int_0^{\frac{\pi}{2}} \sin\theta\,\mathrm{d}\theta}$$

$$= \frac{R \times \frac{1}{2}\int_0^{\frac{\pi}{2}} \sin\theta\,\mathrm{d}\theta}{[-\cos\theta]^{\frac{\pi}{2}}}$$

$$= \frac{1}{4}R\frac{[-\cos 2\theta]^{\frac{\pi}{2}}}{1}$$

$$= \frac{1}{4}R[1-(-1)]$$

$$= \frac{R}{2}$$

This proof is **not** expected to be known for the examination.